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FORCED AND FREE VIBRATIONS OF RECTANGULAR SANDWICH PLATES WITH PARAMETRIC STIFFNESS MODULATION

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An active control of the resonant vibrations of a rectangular sandwich plate performed by the parametric stiffness modulation is analyzed. The controlled vibrations are those of the dominantly flexural type excited by the transverse force acting at the first resonant frequency of dominantly flexural vibrations. The stiffness modulation is performed at a comparatively high frequency identified by the resonance of a mode of the dominantly shear type. The method of direct partition of motions is used that predicts an existence of the modal interaction between these two modes of vibrations due to the parametric stiffness modulation. It is shown that such a parametric control can provide a significant shift of the first eigenfrequency of a controlled plate (the one subjected to the stiffness modulation) from its nominal value for an uncontrolled plate. Heavy fluid loading conditions are accounted for as well as the energy dissipation in the material of a plate. It is demonstrated that although heavy fluid loading reduces resonant frequencies of forced vibrations, the suggested mechanism of control remains valid in these cases. Dynamics of an elementary two-degree-of-freedom model mechanical system is considered to illustrate the mechanism of modal interaction, which is involved in the suggested way of an active control of vibrations of sandwich plates.

1. INTRODUCTION

In many technical applications, it is necessary to control the amplitudes of forced vibrations of thin-walled structures, e.g., plates and shells. In most of the cases, such a control is closely linked to the control of the sound radiation from a vibrating structure and is aimed at improvements in noise, vibration and harshness (NVH) characteristics of cars, aircraft, turbine engines, etc. It is quite typical then that a spectrum of driving forces is given and not subjected to possible modifications. Thus, a reduction in the amplitudes of the structural forced response may be achieved by some "protection from vibrations" of the structure itself. A very detailed description of the up-to-date state of affairs in the control of vibrations is given in reference [1]. Besides a thorough discussion of the passive control of vibrations, which is performed by absorption of the vibrational energy by dissipative elements, the concept of active control is outlined and exemplified in reference [1]. As is seen, feedback and feed forward active controls are considered as the most efficient tools to suppress vibrations. Specifically, the adaptive active control based on measurements of vibration/sound radiation field from primary sources with the use of

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some "corrective" secondary ones is widely explored. In fact, such a control strategy relies upon the linearity of a problem, which justifies use of the superposition principle in forced vibrations of a controlled structure.

An alternative way of active control of vibrations and radiation of sound is associated with the recent advances in material technology, which makes it possible to manufacture the so-called smart materials. In particular, the concept of "dynamic materials" as a special type of smart materials designed to suppress vibrations has been suggested and elaborated in recent papers [2–6]. This concept is in effect based on the ideas of "vibrational rheology" suggested in reference [7]. If a micro-inhomogeneous composite material is considered, then its mechanical "macro-level" characteristics are usually derived by homogenization (averaging) at the micro-level. There are many publications related to various averaging techniques for periodic arrays of elementary cells composing micro-inhomogeneous material, see for example reference [8]. It is not the aim of the present paper to present a detailed discussion of this issue. We just note that in the literature these elementary cells are considered as immobile so that the "global" mechanical properties of a smart material are uniquely defined. However, in principle it is possible to introduce some actuators, which provoke the motions (oscillations) of cells in a prescribed manner. For example, high-frequency small-amplitude vibrations may be excited in these elements of a micro-structure by piezo-electric actuators [1, 9]. Then it appears that the "global" mechanical properties of a smart material after homogenization depend on parameters of these "hidden motions" (e.g., the frequency and the amplitude) besides "static" parameters of cell elements (e.g., dimensions of cells, their material properties, etc.).

In recent papers [2–4] it has been shown that parametric control introduced as the modulation of the stiffness parameters of a structure in response to specific excitation conditions effectively prevents vibrations with large amplitudes. Due to such a parametric control very small stiffness modulation at a certain frequency extinguishes the resonant behaviour of a structure in given excitation conditions. In addition, it is shown in reference [2] that suppression of propagation of flexural waves in an infinitely long plate may also be performed by the same technique.

In the present paper, the concept of parametric stiffness modulation is applied for the analysis of suppression of resonant vibrations of a rectangular sandwich plate in the framework of a standard theory of sandwich plates [10, 11]. A Hamiltonian of the vibrating plate is formulated so that the system of differential equations of its motion and the boundary conditions are derived from the stationarity conditions of this functional. A set of simple Navier boundary conditions is selected and modal analysis of motions is performed firstly for an uncontrolled plate. Then the method of direct partition of motions [7] is used to derive equations of "fast" and "slow" motions of a plate with parametric stiffness modulation. The problem of free and forced vibrations for an undamped plate is solved and the influence of parameters of "vibrational rheology" upon the first resonant frequency of dominantly flexural vibrations is analyzed. Then amplitude response curves are plotted for the cases of forced vibrations with and without material losses and the level of vibration suppression by means of the parametric stiffness control is estimated. The problem of vibrations of a plate in heavy fluid loading conditions is solved and the efficiency of the proposed control method is compared for a fluid-loaded plate and for a plate without fluid loading. To illustrate the mechanism of the proposed method of active control of vibrations, an elementary two-degree-of-freedom system with the parametric stiffness modulation is considered. Two methods—the method of multiple scales and the method of direct partition of motion—give identical results, which

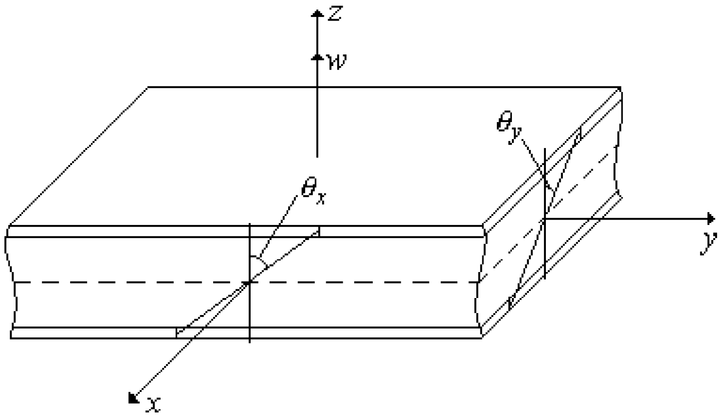

Figure 1. The element of a sandwich plate.

qualitatively explains the effect of suppression of vibrations by the parametric stiffness modulation.

2. EIGENFREQUENCIES AND EIGENMODES OF VIBRATIONS OF A SANDWICH RECTANGULAR PLATE WITHOUT STIFFNESS MODULATION

Consider a plate of sandwich composition, which consists of two symmetrical, relatively thin, stiff skin plies and a thick, soft core ply, see Figure 1. All plies are assumed to be isotropic and the following non-dimensional parameters are introduced to describe the internal structure of a sandwich plate: $\varepsilon = h_{skin}/h_{core}$ as a thickness parameter, $\delta = \rho_{core}/\rho_{skin}$ as a density parameter and $\gamma = E_{core}/E_{skin}$ as a stiffness parameter. Hereafter, subscripts denoting parameters of skin plies are omitted. The length of a plate in x direction is denoted as l_x, in the y direction it is l_y.

A theory of sandwich plate formulated by Skvortsov [11], which has been used to analyze vibrations of sandwich beams (or plates in cylindrical bending) in references [3, 12–13] is adotped. In the framework of this theory, the deformation of a sandwich plate element is governed by three independent variables: a displacement of the mid-surface of the whole element w (which is the same for all plies), a shear angle ϑ_x about the y axis and a shear angle ϑ_y about the x axis.

Derivation of the governing equations of such a plate is easily performed by the use of Hamilton principle and the energy functional is taken as (see reference [3] for a case of cylindrical bending)

$$H = \frac{1}{2}\int_{t_1}^{t_2}\int_0^{l_y}\int_0^{l_x}\left[m\left(\frac{\partial w}{\partial t}\right)^2 + I_1\left(\frac{\partial^2 w}{\partial t\partial x}\right)^2 + I_1\left(\frac{\partial^2 w}{\partial t\partial y}\right)^2 + I_2\left(\frac{\partial \vartheta_x}{\partial t}\right)^2\right.$$

$$\left. + I_2\left(\frac{\partial \vartheta_x}{\partial t}\right)^2 - E_{st}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

$$E_{st} = D_1\left\{(\nabla^2 w)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}$$

$$+D_2\left(\frac{\partial\vartheta_x}{\partial x}\right)^2+D_2\left(\frac{\partial\vartheta_y}{\partial y}\right)^2+vD_2\frac{\partial\vartheta_x}{\partial x}\frac{\partial\vartheta_y}{\partial y}+C_2\left(\frac{\partial\vartheta_x}{\partial x}\frac{\partial\vartheta_y}{\partial y}+\frac{\partial\vartheta_x}{\partial y}\frac{\partial\vartheta_y}{\partial x}\right)$$
$$+\Gamma\left(\frac{\partial w}{\partial x}+\vartheta_x\right)^2+\Gamma\left(\frac{\partial w}{\partial y}+\vartheta_y\right)^2 \qquad (1)$$

Stationarity conditions for functional (1) are formulated as equations of motions

$$-m\frac{\partial^2 w}{\partial t^2}+I_1\frac{\partial^4 w}{\partial t^2\partial x^2}+I_1\frac{\partial^4 w}{\partial t^2\partial y^2}-\frac{\partial^2}{\partial x^2}\left(D_1\frac{\partial^2 w}{\partial x^2}\right)-v\frac{\partial^2}{\partial x^2}\left(D_1\frac{\partial^2 w}{\partial y^2}\right)-v\frac{\partial^2}{\partial y^2}\left(D_1\frac{\partial^2 w}{\partial x^2}\right)$$
$$-\frac{\partial^2}{\partial y^2}\left(D_1\frac{\partial^2 w}{\partial y^2}\right)-4\frac{\partial^2}{\partial x\partial y}\left(C_1\frac{\partial^2 w}{\partial x\partial y}\right)+\frac{\partial}{\partial x}(\Gamma\vartheta_x)+\frac{\partial}{\partial y}(\Gamma\vartheta_y)+\frac{\partial}{\partial x}\left(\Gamma\frac{\partial w}{\partial x}\right)++\frac{\partial}{\partial y}\left(\Gamma\frac{\partial w}{\partial y}\right)=0,$$
$$-I_2\frac{\partial^2\vartheta_x}{\partial t^2}+D_2\frac{\partial^2\vartheta_x}{\partial x^2}+vD_2\frac{\partial^2\vartheta_y}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_y}{\partial x\partial y}+\frac{\partial^2\vartheta_x}{\partial y^2}\right)-\Gamma\left(\vartheta_x+\frac{\partial w}{\partial x}\right)=0,$$
$$-I_2\frac{\partial^2\vartheta_y}{\partial t^2}+D_2\frac{\partial^2\vartheta_y}{\partial y^2}+vD_2\frac{\partial^2\vartheta_x}{\partial x\partial y}+C_2\left(\frac{\partial^2\vartheta_x}{\partial x\partial y}+\frac{\partial^2\vartheta_y}{\partial x^2}\right)-\Gamma\left(\vartheta_y+\frac{\partial w}{\partial y}\right)=0. \qquad (2)$$

The effective elastic parameters are

$$D_1=\frac{Eh^3}{12(1-v^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right),\quad C_1=\frac{Eh^3}{12(1-v^2)}\left(2+\frac{\gamma}{\varepsilon^3}\right)\frac{(1-v)}{2},$$
$$D_2=\frac{Eh^3}{2(1-v^2)}\left(1+\frac{1}{\varepsilon}\right)^2,\quad C_2=\frac{Eh^3}{2(1-v^2)}\left(1+\frac{1}{\varepsilon}\right)^2\frac{(1-v)}{2},$$
$$\Gamma=\frac{Eh}{(1-v^2)}\frac{1-v}{2}\varepsilon\gamma\left(1+\frac{1}{\varepsilon}\right)^2,$$
$$I_1=\frac{\rho h^3}{12}\left(2+\frac{\delta}{\varepsilon^3}\right),\quad I_2=\frac{\rho h^3}{2}\left(1+\frac{1}{\varepsilon}\right)^2,\quad m=\rho h\left(2+\frac{\delta}{\varepsilon}\right). \qquad (3)$$

The formulation of arbitrary boundary conditions in the case of one-dimensional cylindrical bending of a sandwich plate is given in reference [12]. Its generalization for two-dimensional case is straightforward and is not displayed here because only a simply supported plates considered:

$$x=0, x=l_x:\quad w=0,\quad M_{1x}\equiv-\frac{\partial^2 w}{\partial x^2}-v\frac{\partial^2 w}{\partial y^2}=0,\quad M_{2x}\equiv\frac{\partial\vartheta_x}{\partial x}+v\frac{\partial\vartheta_y}{\partial y}=0,$$
$$y=0, y=l_x:\quad w=0,\quad M_{1y}\equiv-\frac{\partial^2 w}{\partial y^2}-v\frac{\partial^2 w}{\partial x^2}=0,\quad M_{2y}\equiv\frac{\partial\vartheta_y}{\partial y}+v\frac{\partial\vartheta_x}{\partial x}=0. \qquad (4)$$

Before addressing the vibrations of a plate with the modulated stiffness it is necessary to find spectra of eigenfrequencies of a plate with the constant stiffness. To satisfy boundary conditions (4) the lateral displacement and the shear angle are sought in expansions of trigonometric functions:

$$w(x,y,t)=\sum_k\sum_m W_{s11}\sin\left(\frac{k\pi x}{l_x}\right)\sin\left(\frac{m\pi y}{l_y}\right)\exp(-\mathrm{i}\omega t),$$
$$\vartheta_x(x,y,t)=\sum_k\sum_m \Theta_{sx11}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right)\exp(-\mathrm{i}\omega t),$$
$$\vartheta_y(x,y,t)=\sum_k\sum_m \Theta_{sy11}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right)\exp(-\mathrm{i}\omega t). \qquad (5)$$

It is clear that depending on the combination of parameters (k, m) one obtains either a symmetrical mode of vibrations or a skew-symmetrical mode. It is convenient to introduce a non-dimensional frequency parameter as $\omega h/c_p$, with $c_p = \sqrt{E/\rho(1-\nu^2)}$ as a sound speed in the material of the skin plies of a sandwich plate. In the case of a simply supported plate, eigenfrequencies $\omega_{kmj}, j = 1, 2, 3$ for each combination of numbers k and m are found from an individual bi-cubic equation:

$$\begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = 0. \qquad (6)$$

The elements of this determinant $d_{kn} \equiv d_{kn}(k, m, \omega_{kmj}h/c, \delta, \gamma, \varepsilon)$ are elementary formulated via coefficients (3). They are rather cumbersome and are not presented here for brevity.

The lateral displacement w and the shear angles ϑ_x, ϑ_y are involved in vibrations of sandwich plates. To judge which component of the motions is dominant in free vibrations at the given eigenfrequency ω_{kmj}, it is necessary to find the relevant modal coefficients A_{kmj}, B_{kmj}, C_{kmj}. If one sets $C_{kmj} \equiv 1$, then the other two modal coefficients for each ω_{kmj} are calculated as

$$A_{kmj} = -\begin{vmatrix} d_{13} & d_{12} \\ d_{23} & d_{22} \end{vmatrix} \Bigg/ \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}, \quad B_{kmj} = -\begin{vmatrix} d_{11} & d_{13} \\ d_{21} & d_{23} \end{vmatrix} \Bigg/ \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}. \qquad (7)$$

Dominantly flexural vibrations are defined as those having modal coefficient A_{kmj} larger than B_{kmj} and C_{kmj}. If it is not the case, then free vibrations are associated with a strong participation of the shear deformations.

In Figure 2(a), the frequency parameter $\omega_{km1}h/c_p$ for the first eigenfrequency of a square plate $(l_y/l_x = 1)$ found from equation (6) is plotted versus parameter l_x/h by curves 1 and 2 for $k = 1, m = 1$ and $k = 2, m = 2$ respectively. These two graphs are typical for other values of a shape parameter l_y/l_x. The set of principal parameters of a sandwich plate composition is taken as $\delta = 0{\cdot}1$, $\gamma = \gamma_0 = 0{\cdot}01$, $\varepsilon = 0{\cdot}05$ and this set is standard for naval structures. The dependence of relevant modal coefficients A_{kmj} and B_{kmj} $(C_{kmj} \equiv 1)$ on parameters l_x/h and l_y/h is shown in Figure 2(b,c) (it is almost the same for $k = 1, m = 1$ and $k = 2, m = 2$). As is seen, for all values of l_x/h and l_y/h, the modulus of the former one is markedly larger than 1, whereas the latter one is close to 1. Thus, these eigenfrequencies are attributed to the spectrum of dominantly flexural vibrations. In Figure 3(a), the frequency parameter $\omega_{km2}h/c_p$ for the second eigenfrequency of a square plate $(l_y/l_x = 1)$ found from equation (6) is plotted versus parameter l_x/h similar to Figure 2(a). The modal coefficient A_{km2} equals zero, the dependence of B_{km2} $(C_{km2} \equiv 1)$ on parameters l_x/h and l_y/h is shown in Figure 3b. For all values of l_x/h and l_y/h, these eigenfrequencies are attributed to the spectrum of purely shear vibrations. Similar graphs plotted for eigenfrequencies of the third spectrum (see Figure 4(a–c)) suggest that resonant vibrations at these frequencies are of dominantly shear type since the modal coefficients A_{km3} and B_{km3} are of order 1. Graphs shown in Figures 2–4 are typical for sandwich plates having other realistic geometry and stiffness parameters. Similar to the cases considered in the references [3, 4], it is important to notice that eigenfrequencies of the third (dominantly shear) spectrum are much higher than the relevant eigenfrequencies from the first (dominantly flexural) spectrum. This makes it possible to control low-frequency resonant flexural vibrations by the parametric stiffness modulation performed at a comparatively high frequency of resonant shear vibrations.

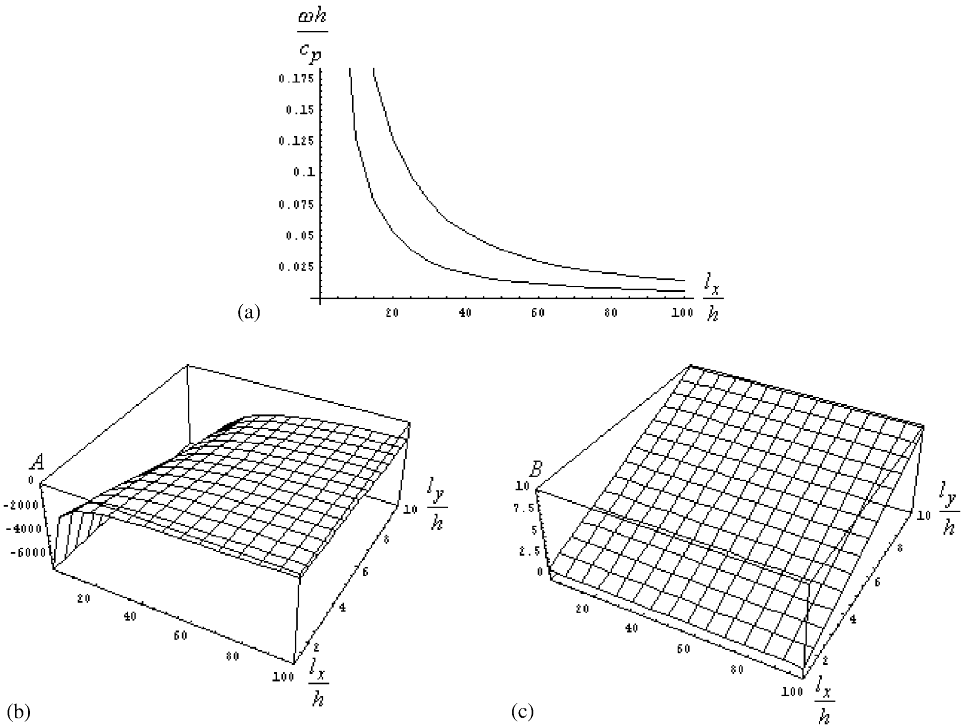

Figure 2. (a) The first eigenfrequency parameter $\omega_{mk1}h/c_p$ of a square plate versus l_x/h. (b) The modal coefficient A_{km1} versus l_x/h and l_y/h. (c) The modal coefficient B_{km1} versus l_x/h and l_y/h.

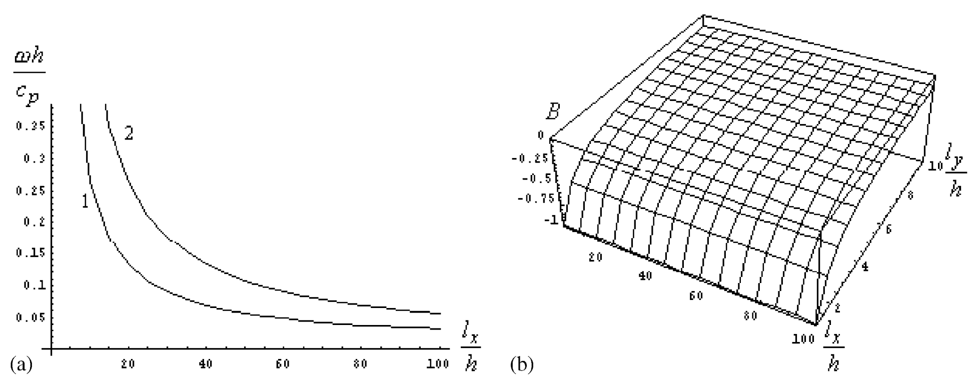

Figure 3. (a) The second eigenfrequency parameter $\omega_{mk2}h/c_p$ of a square plate versus l_x/h. (b) The modal coefficient B_{km3} versus l_x/h and l_y/h.

3. EQUATIONS OF VIBRATIONS OF A RECTANGULAR SANDWICH PLATE WITH PARAMETRIC STIFFNESS MODULATION

Consider forced vibrations of a sandwich plate generated by a lateral force acting at a frequency ω_s assumed to be fairly close to the first resonant frequency of dominantly

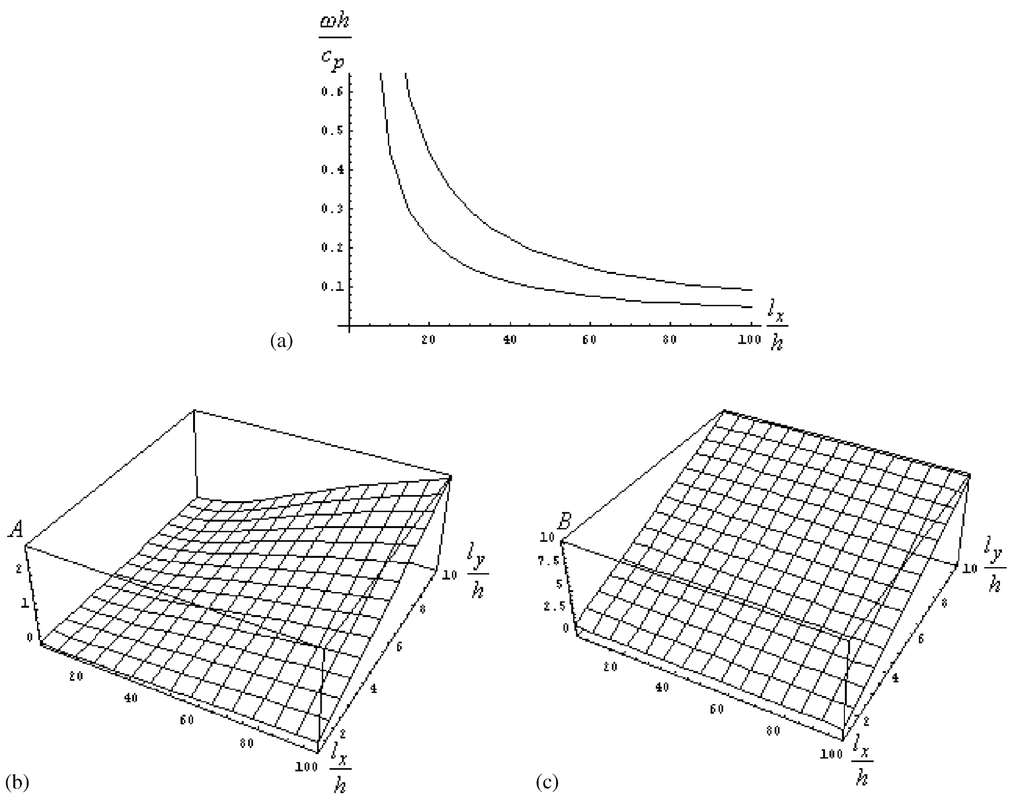

Figure 4. (a) The third eigenfrequency parameter $\omega_{mk3}h/c_p$ of a square plate versus l_x/h. (b) The modal coefficient A_{km3} versus l_x/h and l_y/h. (c) The modal coefficient B_{km3} versus l_x/h and l_y/h.

flexural vibrations,

$$q(x, y, t) = q_0(x, y)\cos \omega_q t \quad \omega_q \approx \omega_{111}. \tag{8}$$

In the case of such excitation, the forced response of a sandwich plate should be of resonant type. In order to avoid large amplitudes of vibrations, an active control mechanism is used, which is based on the concept of "vibrational rheology" suggested by Blekhman [7]. Since the frequency of a driving force is given, the only possibility to suppress the resonant behaviour is associated with a shift of the given eigenfrequency of an uncontrolled plate from its "nominal" value. Such a shift is achievable by means of the parametric stiffness modulation both in time and in space domains. Let the stiffness coefficient γ be decomposed into the sum of the "bulk" constant component γ_0 and some fairly small fluctuating component $\gamma_1(x, y, t)$, i.e.,

$$\gamma(x, y, t) = \gamma_0 + \gamma_1(x, y, t). \tag{9}$$

From the practical viewpoint, it is unrealistic to arrange large modulations of the stiffness, so the following inequality is held:

$$|\gamma_1(x, y, t)| \ll \gamma_0. \tag{10}$$

Then equations of motions (2) are formulated as

$$-\frac{1}{c_p^2}\left(2+\frac{\delta}{\varepsilon}\right)\left[\frac{\partial^2 w}{\partial t^2}+\frac{h^2}{12}\frac{\partial^4 w}{\partial t^2\partial x^2}+\frac{h^2}{12}\frac{\partial^4 w}{\partial t^2\partial y^2}\right]-\frac{h^2}{12}\left\{\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{(\gamma_0+\gamma_1)}{\varepsilon^3}\right)\frac{\partial^2 w}{\partial x^2}\right]\right.$$

$$+\nu\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{(\gamma_0+\gamma_1)}{\varepsilon^3}\right)\frac{\partial^2 w}{\partial y^2}\right]+\nu\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{(\gamma_0+\gamma_1)}{\varepsilon^3}\right)\frac{\partial^2 w}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{(\gamma_0+\gamma_1)}{\varepsilon^3}\right)\frac{\partial^2 w}{\partial y^2}\right]$$

$$\left.-2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[\left(2+\frac{(\gamma_0+\gamma_1)}{\varepsilon^3}\right)\frac{\partial^2 w}{\partial x\partial y}\right]\right\}+\frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial x}\left[(\gamma_0+\gamma_1)\left(\vartheta_x+\frac{\partial w}{\partial x}\right)\right]$$

$$+\frac{1-\nu}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial y}\left[(\gamma_0+\gamma_1)\left(\vartheta_y+\frac{\partial w}{\partial y}\right)\right]+\frac{q(1-\nu^2)}{Eh}=0,$$

$$-\frac{1}{c_p^2}\frac{\partial^2\vartheta_x}{\partial t^2}+\frac{\partial^2\vartheta_x}{\partial x^2}+\frac{(1+\nu)}{2}\frac{\partial^2\vartheta_y}{\partial x\partial y}+\frac{(1-\nu)}{2}\frac{\partial^2\vartheta_x}{\partial y^2}-\frac{(1-\nu)}{2}\frac{\varepsilon(\gamma_0+\gamma_1)}{h^2}\left(\vartheta_x+\frac{\partial w}{\partial x}\right)=0,$$

$$-\frac{1}{c_p^2}\frac{\partial^2\vartheta_y}{\partial t^2}+\frac{\partial^2\vartheta_y}{\partial y^2}+\frac{(1+\nu)}{2}\frac{\partial^2\vartheta_x}{\partial x\partial y}+\frac{(1-\nu)}{2}\frac{\partial^2\vartheta_y}{\partial x^2}-\frac{(1-\nu)}{2}\frac{\varepsilon(\gamma_0+\gamma_1)}{h^2}\left(\vartheta_y+\frac{\partial w}{\partial y}\right)=0. \quad (11)$$

The stiffness modulation is assumed to be a periodic function of a rather high frequency ω_m,

$$\gamma_1(x,y,t)=\bar{\gamma}_1(x,y)\cos\omega_m t \qquad \omega_m \gg \omega_q. \qquad (12, 13)$$

Inequality (13) permits one to apply the method of direct partition of motions. As there are two different time scales in the input into the sandwich plate, it is reasonable to assume the response to be decomposed as

$$\begin{aligned} w(x,y,t) &= w_s(x,y,t)+w_f(x,y,t),\\ \vartheta_x(x,y,t) &= \vartheta_{sx}(x,y,t)+\vartheta_{fx}(x,y,t),\\ \vartheta_y(x,y,t) &= \vartheta_{sy}(x,y,t)+\vartheta_{fy}(x,y,t). \end{aligned} \qquad (14)$$

In formulas (14), the first letter in the subscripts designates "slow" and "fast" components of motions respectively. To solve this system of partial differential equations, we use condition (13) and formulate "fast" components as those dependent upon two different time scales ($\omega_f \equiv \omega_m$):

$$\begin{aligned} w_f(x,y,t) &= \tilde{w}_f(x,y,t)\cos\omega_f t,\\ \vartheta_{fx}(x,y,t) &= \tilde{\vartheta}_{fx}(x,y,t)\cos\omega_f t,\\ \vartheta_{fy}(x,y,t) &= \tilde{\vartheta}_{fy}(x,y,t)\cos\omega_f t. \end{aligned} \qquad (15)$$

In a sense (see similar discussion in reference [3]), this assumption is equivalent to the approach, which is conveniently used in the method of multiple scales, see reference [14]. Indeed, two time variables are introduced to describe the motion of a system, i.e., $w(x,y,t)=w_0(x,y,T_0,T_1,\ldots)+\varepsilon_0 w_1(w,y,T_0,T_1,\ldots)+\cdots$ with $T_0=t$, $T_1=\varepsilon_0 t,\ldots,\varepsilon_0$ is a formal small parameter. The function $\cos\omega_f t$ in equation (15) displays fast oscillations whilst the functions $\tilde{w}_f(x,y,t)$, $\tilde{\vartheta}_{fx}(x,y,t)$, $\tilde{\vartheta}_{fy}(x,y,t)$ define slowly modulated amplitudes of these oscillations. The comparison of results obtained by the method of multiple scales and the method of direct partition of motions is addressed in more detail in section 8.

The inequality $\omega_s \ll \omega_f$ is used to formulate inertial terms in equations (2) asympotically as

$$\begin{aligned} \ddot{w}_f(x,y,t) &\approx -\omega_f^2 \tilde{w}_f(x,y,t)\cos\omega_f t, \\ \ddot{\vartheta}_{fx}(x,y,t) &\approx -\omega_f^2 \tilde{\vartheta}_{fx}(x,y,t)\cos\omega_f t, \\ \ddot{\vartheta}_{fy}(x,y,t) &\approx -\omega_f^2 \tilde{\vartheta}_{fy}(x,y,t)\cos\omega_f t. \end{aligned} \tag{16}$$

Formulas (14)–(16) are substituted in to equations (2) and all the terms containing $\cos\omega_f t$ are collected to yield

$$\begin{aligned} &\frac{\omega_f^2}{c_p^2}\left(2+\frac{\delta}{\varepsilon}\right)\left[\overleftrightarrow{w}_f - \frac{h^2}{12}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x^2} - \frac{h^2}{12}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial y^2}\right] - \frac{h^2}{12}\left\{\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x^2}\right]\right. \\ &+ \nu\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 \overleftrightarrow{w}_f}{\partial y^2}\right] + \nu\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x^2}\right] + \frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 \overleftrightarrow{w}_f}{\partial y^2}\right] \\ &\left.-2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x\partial y}\right]\right\} - \frac{h^2}{12}\left\{\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 w_s}{\partial x^2}\right]\right. \\ &\left.+ \nu\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 w_s}{\partial y^2}\right] + \nu\frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 w_s}{\partial x^2}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 w_s}{\partial y^2}\right] - 2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 w_s}{\partial x\partial y}\right]\right\} \\ &+ \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\tilde{\vartheta}_{fx}+\frac{\partial \tilde{w}_f}{\partial x}\right)+\gamma_1\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right)\right] \\ &+ \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\tilde{\vartheta}_{fy}+\frac{\partial \tilde{w}_f}{\partial y}\right)+\gamma_1\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right)\right] \\ &+ \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\tilde{\vartheta}_{fx}+\frac{\partial \tilde{w}_f}{\partial x}\right)+\gamma_1\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right)\right] \\ &+ \frac{(1-\nu)}{2}\left(1+\frac{1}{\varepsilon}\right)^2 \varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\tilde{\vartheta}_{fy}+\frac{\partial \tilde{w}_f}{\partial y}\right)+\gamma_1\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right)\right] = 0, \end{aligned}$$

$$\begin{aligned} &\frac{\omega_f^2}{c_p^2}\tilde{\vartheta}_{fx} + \frac{\partial^2 \tilde{\vartheta}_{fx}}{\partial x^2} + \frac{(1-\nu)}{2}\frac{\partial^2 \tilde{\vartheta}_{fy}}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 \tilde{\vartheta}_{fx}}{\partial y^2} \\ &-\frac{(1-\nu)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\tilde{\vartheta}_{fx}+\frac{\partial \tilde{w}_f}{\partial x}\right) - \frac{(1-\nu)}{2}\frac{\varepsilon\gamma_1}{h^2}\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right) = 0, \\ &\frac{\omega_f^2}{c_p^2}\tilde{\vartheta}_{fy} + \frac{\partial^2 \tilde{\vartheta}_{fy}}{\partial y^2} + \frac{(1-\nu)}{2}\frac{\partial^2 \tilde{\vartheta}_{fx}}{\partial x\partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 \tilde{\vartheta}_{fy}}{\partial x^2} \\ &-\frac{(1-\nu)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\tilde{\vartheta}_{fy}+\frac{\partial \tilde{w}_f}{\partial y}\right) - \frac{(1-\nu)}{2}\frac{\varepsilon\gamma_1}{h^2}\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right) = 0. \end{aligned} \tag{17}$$

These equations are subtracted from the original equations (2) and in accordance with the procedure of the direct partition of motions (see reference [7]), averaging over a period of fast motions is performed to yield

$$
\begin{aligned}
&-\frac{1}{c_p^2}\left(2+\frac{\delta}{\varepsilon}\right)\left[\frac{\partial^2 w_s}{\partial t^2}-\frac{h^2}{12}\frac{\partial^4 w_s}{\partial t^2\partial x^2}-\frac{h^2}{12}\frac{\partial^2 w_s}{\partial t^2\partial y^2}\right]-\frac{h^2}{12}\left\{\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 w_s}{\partial x^2}\right]\right.\\
&+v\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 w_s}{\partial y^2}\right]+v\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 w_s}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 w_s}{\partial y^2}\right]\\
&\left.-2(1-v)\frac{\partial^2}{\partial x\partial y}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2 w_s}{\partial x\partial y}\right]\right\}-\frac{h^2}{24}\left\{\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x^2}\right]\right.\\
&\left.+v\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial y^2}\right]+v\frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial y^2}\right]-2(1-v)\frac{\partial^2}{\partial x\partial y}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2 \overleftrightarrow{w}_f}{\partial x\partial y}\right]\right\}\\
&+\frac{(1-v)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right)+\frac{1}{2}\gamma_1\left(\tilde{\vartheta}_{fx}+\frac{\partial \tilde{w}_f}{\partial x}\right)\right]\\
&+\frac{(1-v)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right)+\frac{1}{2}\gamma_1\left(\tilde{\vartheta}_{fy}+\frac{\partial \tilde{w}_f}{\partial y}\right)\right]\\
&+\frac{(1-v)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right)+\frac{1}{2}\gamma_1\left(\tilde{\vartheta}_{fx}+\frac{\partial \tilde{w}_f}{\partial x}\right)\right]\\
&+\frac{(1-v)}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right)+\frac{1}{2}\gamma_1\left(\tilde{\vartheta}_{fy}+\frac{\partial \tilde{w}_f}{\partial y}\right)\right]+\frac{q(1-v^2)}{Eh}=0,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{c_p^2}\frac{\partial^2\vartheta_{sx}}{\partial t^2}+\frac{\partial^2\vartheta_{sx}}{\partial x^2}+\frac{(1+v)}{2}\frac{\partial^2\vartheta_{sy}}{\partial x\partial y}+\frac{(1-v)}{2}\frac{\partial^2\vartheta_{sx}}{\partial y^2}\\
&-\frac{(1-v)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\vartheta_{sx}+\frac{\partial w_s}{\partial x}\right)-\frac{(1-v)}{4}\frac{\varepsilon\gamma_1}{h^2}\left(\tilde{\vartheta}_{fx}+\frac{\partial\tilde{w}_f}{\partial x}\right)=0,\\
&-\frac{1}{c_p^2}\frac{\partial^2\vartheta_{sy}}{\partial t^2}+\frac{\partial^2\vartheta_{sy}}{\partial y^2}+\frac{(1+v)}{2}\frac{\partial^2\vartheta_{sx}}{\partial x\partial y}+\frac{(1-v)}{2}\frac{\partial^2\vartheta_{sy}}{\partial x^2}\\
&-\frac{(1-v)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\vartheta_{sy}+\frac{\partial w_s}{\partial y}\right)-\frac{(1-v)}{4}\frac{\varepsilon\gamma_1}{h^2}\left(\tilde{\vartheta}_{fy}+\frac{\partial\tilde{w}_f}{\partial y}\right)=0.
\end{aligned}
\tag{18}
$$

Equations (18) are linear with respect to displacements and shear angles, so the forced response is now sought as ($\omega_s \equiv \omega_q$)

$$
\begin{aligned}
w_s(x,y,t)&=\hat{w}_s(x,y)\cos\omega_s t,\\
\vartheta_{sx}(x,y,t)&=\hat{\vartheta}_{sx}(x,y)\cos\omega_s t,\\
\vartheta_{sy}(x,y,t)&=\hat{\vartheta}_{sy}(x,y)\cos\omega_s t.
\end{aligned}
\tag{19}
$$

Similarly, due to the linearity of equations (18) the functions defining slow amplitude modulations of fast motions become

$$
\begin{aligned}
\tilde{w}_f(x,y,t)&=\hat{w}_f(x,y)\cos\omega_s t,\\
\tilde{\vartheta}_{fx}(x,y,t)&=\hat{\vartheta}_{fx}(x,y)\cos\omega_s t,\\
\tilde{\vartheta}_{fy}(x,y,t)&=\hat{\vartheta}_{fy}(x,y)\cos\omega_s t.
\end{aligned}
\tag{20}
$$

Thus, we substitute these functions in equations (18), omit time dependence expressed as $\cos \omega_s t$ and obtain a system of three ordinary differential equations for "slow" motions of a sandwich plate:

$$\frac{\omega_2^2}{c_p^2}\left(2+\frac{\delta}{\varepsilon}\right)\left[\hat{w}_s-\frac{h^2}{12}\frac{\partial^2\hat{w}_s}{\partial x^2}-\frac{h^2}{12}\frac{\partial^2\hat{w}_s}{\partial y^2}\right]-\frac{h^2}{12}\left\{\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2\hat{w}_s}{\partial x^2}\right]\right.$$
$$+v\frac{\partial^2}{\partial x^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2\hat{w}_s}{\partial y^2}\right]+v\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2\hat{w}_s}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2\overleftrightarrow{w}_s}{\partial y^2}\right]$$
$$\left.-2(1-v)\frac{\partial^2}{\partial x\partial y}\left[\left(2+\frac{\gamma_0}{\varepsilon^3}\right)\frac{\partial^2\hat{w}_s}{\partial x\partial y}\right]\right\}-\frac{h^2}{24}\left\{\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2\hat{w}_f}{\partial x^2}\right]\right.$$
$$\left.+v\frac{\partial^2}{\partial x^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2\hat{w}_f}{\partial y^2}\right]+v\frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2\hat{w}_f}{\partial x^2}\right]+\frac{\partial^2}{\partial y^2}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2\hat{w}_f}{\partial y^2}\right]-2(1-v)\frac{\partial^2}{\partial x\partial y}\left[\frac{\gamma_1}{\varepsilon^3}\frac{\partial^2\hat{w}_f}{\partial x\partial y}\right]\right\}$$
$$+\frac{1-v}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\hat{\vartheta}_{sx}+\frac{\partial\hat{w}_s}{\partial x}\right)+\frac{1}{2}\gamma_1\left(\hat{\vartheta}_{fx}+\frac{\partial\hat{w}_f}{\partial x}\right)\right]$$
$$+\frac{1-v}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\hat{\vartheta}_{sy}+\frac{\partial\hat{w}_s}{\partial y}\right)+\frac{1}{2}\gamma_1\left(\hat{\vartheta}_{fy}+\frac{\partial\hat{w}_f}{\partial y}\right)\right]$$
$$+\frac{1-v}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial x}\left[\gamma_0\left(\hat{\vartheta}_{sx}+\frac{\partial\hat{w}_s}{\partial x}\right)+\frac{1}{2}\gamma_1\left(\hat{\vartheta}_{fx}+\frac{\partial\hat{w}_f}{\partial x}\right)\right]$$
$$+\frac{1-v}{2}\left(1+\frac{1}{\varepsilon}\right)^2\varepsilon\frac{\partial}{\partial y}\left[\gamma_0\left(\hat{\vartheta}_{sy}+\frac{\partial\hat{w}_s}{\partial y}\right)+\frac{1}{2}\gamma_1\left(\hat{\vartheta}_{fy}+\frac{\partial\hat{w}_f}{\partial y}\right)\right]+\frac{q_0(1-v^2)}{Eh}=0,$$

$$\frac{\omega_s^2}{c_p^2}\hat{\vartheta}_{sx}+\frac{\partial^2\hat{\vartheta}_{sx}}{\partial x^2}+\frac{(1+v)}{2}\frac{\partial^2\hat{\vartheta}_{sy}}{\partial x\partial y}+\frac{(1-v)}{2}\frac{\partial^2\hat{\vartheta}_{sx}}{\partial y^2}$$
$$-\frac{1(-v)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\hat{\vartheta}_{sx}+\frac{\partial\hat{w}_s}{\partial x}\right)-\frac{(1-v)}{4}\frac{\varepsilon\gamma_1}{h^2}\left(\hat{\vartheta}_{fx}+\frac{\partial\hat{w}_f}{\partial x}\right)=0,$$
$$\frac{\omega_s^2}{c_p^2}\hat{\vartheta}_{sy}+\frac{\partial^2\hat{\vartheta}_{sy}}{\partial y^2}+\frac{(1+v)}{2}\frac{\partial^2\hat{\vartheta}_{sx}}{\partial x\partial y}+\frac{(1-v)}{2}\frac{\partial^2\hat{\vartheta}_{sy}}{\partial x^2}$$
$$-\frac{(1-v)}{2}\frac{\varepsilon\gamma_0}{h^2}\left(\hat{\vartheta}_{sy}+\frac{\partial\hat{w}_s}{\partial y}\right)-\frac{(1-v)}{4}\frac{\varepsilon\gamma_1}{h^2}\left(\hat{\vartheta}_{fy}+\frac{\partial\hat{w}_f}{\partial y}\right)=0. \qquad (21)$$

The equations of "fast" motions are written as equation (17) provided that the set of "slow" functions $(w_s, \vartheta_{sx}, \vartheta_{sy})$ is replaced by $(\hat{w}_s, \hat{\vartheta}_{sx}, \hat{\vartheta}_{sy})$ and the set of "fast" functions $(\tilde{w}_f, \tilde{\vartheta}_{fx}, \tilde{\vartheta}_{fy})$ is replaced by $(\hat{w}_f, \hat{\vartheta}_{fx}, \hat{\vartheta}_{fy})$. As seen from equations (17) and (21), the coupling of "slow" and "fast" motions is generated by the function $\bar{\gamma}_1(x, y)$. If this function is set to zero, then as expected the forced vibrations at the frequency $\omega_q \equiv \omega_s$ are not influenced by the free vibrations at the frequency $\omega_m \equiv \omega_f$. Here, that the coupling of equations of "slow" and "fast" motions could be described in this rather straightforward manner because we have assumed inequality (13) to be valid. As soon as the modulation frequency becomes insufficiently "high" as compared with the forcing frequency, the coupling becomes more complicated. This aspect is not presented any further because the careful analysis of applicability limits for the method of direct partition of motions has been performed by several authors, see, for example reference [15].

4. MODAL ANALYSIS OF FORCED VIBRATIONS OF A RECTANGULAR SANDWICH PLATE WITH PARAMETRIC STIFFNESS MODULATION

Consider now the forced resonant response of a sandwich plate to an external excitation. The spatial distribution of a driving force is for simplicity taken to be of the shape of the first resonant mode, i.e.,

$$q(x,y,t) = q_0(x,y)\cos\omega_{111}t = Q\sin\frac{\pi x}{l_x}\sin\frac{\pi y}{l_y}\cos\omega_{111}t. \tag{22}$$

The forced vibrations of a plate with parametric stiffness modulation are sought by the Galerkin method in the form of four-term expansions of the functions, which exactly satisfy all boundary conditions:

$$\begin{aligned}
\hat{w}_s(x,y) &= W_{s11}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + W_{s12}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right)\\
&\quad + W_{s21}\sin\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + W_{s22}\sin\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right),\\
\vartheta_{sx}(x,y) &= \Theta_{sx11}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + \Theta_{sx12}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right)\\
&\quad + \Theta_{sx21}\cos\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + \Theta_{sx22}\cos\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right),\\
\vartheta_{sy}(x,y) &= \Theta_{sy11}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right) + \Theta_{sy12}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{2\pi y}{l_y}\right)\\
&\quad + \Theta_{sy21}\sin\left(\frac{2\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right) + \Theta_{sy22}\sin\left(\frac{2\pi x}{l_x}\right)\cos\left(\frac{2\pi y}{l_y}\right),
\end{aligned} \tag{23}$$

$$\begin{aligned}
\hat{w}_f(x,y) &= W_{f11}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + W_{f12}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right)\\
&\quad + W_{f21}\sin\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + W_{f22}\sin\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right),\\
\vartheta_{fx}(x,y) &= \Theta_{fx11}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + \Theta_{fx12}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right)\\
&\quad + \Theta_{fx21}\cos\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) + \Theta_{fx22}\cos\left(\frac{2\pi x}{l_x}\right)\sin\left(\frac{2\pi y}{l_y}\right),\\
\vartheta_{fy}(x,y) &= \Theta_{fy11}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right) + \Theta_{fy12}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{2\pi y}{l_y}\right)\\
&\quad + \Theta_{fy21}\sin\left(\frac{2\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right) + \Theta_{fy22}\sin\left(\frac{2\pi x}{l_x}\right)\cos\left(\frac{2\pi y}{l_y}\right).
\end{aligned} \tag{24}$$

This approximation is sufficient to describe the effect of the stiffness modulation. As has been discussed in the previous section, the trial functions used in equation (23) and (24) constitute the set of eigenmodes of an uncontrolled plate.

Since the excitation is specified as equation (22) it is obvious that the amplitude W_{s11} tends to infinity when $\bar{\gamma}_1(x,y) \equiv 0$, i.e., when a sandwich plate performs resonant vibrations at its first eigenfrequency. To reduce the amplitude of forced vibrations of a plate, the mechanism of an active control, which is based on a modal interaction due to the parametric stiffness modulation is suggested. If expansions (23) and (24) are substituted into equations (17) and (21), then it may be seen that a proper choice of the spatial distribution of the stiffness modulation (specified by the function $\bar{\gamma}_1(x, y)$) introduces the modal coupling between various modes. As the driving force is chosen as equation (22) and therefore the first eigenmode is resonantly excited, it is necessary to select the function $\bar{\gamma}_1(x, y)$ to couple this mode to another one, which has its eigenfrequency much higher than ω_{111}. Apparently, there is not a unique choice, but the simple approach is to couple the directly excited plate's motions at the frequency ω_{111} with the motions at the frequency ω_{223}, which, as has been shown in the previous section, obeys the inequality (13). It is possible if the function $\bar{\gamma}_1(x, y)$ is selected as

$$\bar{\gamma}_1(x, y) = \hat{\gamma}_1 \cos\frac{\pi x}{l_x}\cos\frac{\pi y}{l_y}. \tag{25}$$

Formulas (23)–(25) are substituted into equations (17) and (21) and Galerkin's procedure is applied, the system of differential equations is transformed to the system of 24 linear algebraic equations with respect to the amplitudes introduced in formulas (23)–(34)

Since a spatial distribution of the stiffness modulation is given by formula (25), this algebraic system is split into two sub-systems due to orthonormal properties of trigonometric functions. One of these sub-systems is formulated with respect to odd "fast" and "slow" modes (i.e., the modes with $(k = 1, m = 2)$ and $(k = 2, m = 1)$). These modes are neither directly excited nor coupled with the modes $(k = 1, m = 1)$, $(k = 2, m = 2)$ and their amplitudes vanish. Another system is formulated with respect to even "fast" and "slow" modes (i.e., the modes with $(k = 1, m = 1)$ and $(k = 2, m = 2)$ and in turn is sub-divided into two sub-sub-systems. The first one formulates a link between "slow" symmetric motions $(k = 1, m = 1)$ and "fast" skew-symmetric motions $(k = 2, m = 2)$. In contrast, the second one contains amplitudes of "fast" symmetric motions and "slow" skew-symmetric motions. The possibility of suppressing the resonant vibrations at the first resonant frequency of flexural vibrations is considered. To deal with this case, it is necessary to consider the first sub-sub-system of equations formulated as

$$\begin{pmatrix} a_{11} - b_1\omega_s^2 & a_{12} & a_{13} & \hat{\gamma}_1 a_{14} & \hat{\gamma}_1 a_{15} & \hat{\gamma}_1 a_{16} \\ a_{21} & a_{22} - b_2\omega_s^2 & a_{23} & \hat{\gamma}_1 a_{24} & \hat{\gamma}_1 a_{25} & \hat{\gamma}_1 a_{26} \\ a_{31} & a_{32} & a_{33} - b_3\omega_s^2 & \hat{\gamma}_1 a_{34} & \hat{\gamma}_1 a_{35} & \hat{\gamma}_1 a_{23} \\ \hat{\gamma}_1 a_{41} & \hat{\gamma}_1 a_{42} & \hat{\gamma}_1 a_{43} & a_{44} - b_4\omega_f^2 & a_{45} & a_{46} \\ \hat{\gamma}_1 a_{51} & \hat{\gamma}_1 a_{52} & \hat{\gamma}_1 a_{53} & a_{54} & a_{55}{-}b_5\omega_f^2 & a_{56} \\ \hat{\gamma}_1 a_{61} & \hat{\gamma}_1 a_{62} & \hat{\gamma}_1 a_{63} & a_{64} & a_{65} & a_{66} - b_6\omega_f^2 \end{pmatrix} \begin{pmatrix} W_{s11} \\ \Theta_{sx11} \\ \Theta_{sy11} \\ W_{f22} \\ \Theta_{fx22} \\ \Theta_{fy22} \end{pmatrix} = \begin{pmatrix} \hat{Q} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{26}$$

It relates amplitudes of displacements $W_{s11}, \Theta_{sx11}, \Theta_{sy11}, W_{f22}, \Theta_{fx22}, \Theta_{fy22}$ to the amplitude $\hat{Q} = Q(1 - \nu^2)l_x/Eh$ of a driving force applied at the excitation frequency ω_s. Apparently, when excitation frequency ω_s is fairly close to the first resonant frequency of a plate without control $(\gamma_1 = 0)$, the quantity W_{s11} becomes very large. To reduce the amplitude of this component of displacements, the stiffness modulation is applied.

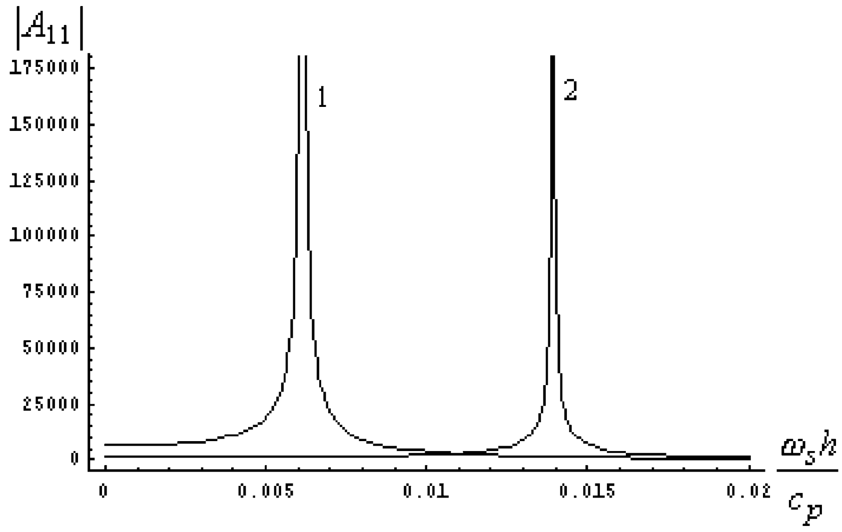

Figure 5. The amplitude W_{s11} versus frequency parameter $\omega_s h/c_p$.

In Figure 5, a dependence of the amplitude $W_{s11} \equiv A_{11}$ on frequency parameter $\omega_s h/c_p$ is plotted for the following set of parameters of a sandwich plate composition: $\delta = 0{\cdot}1$, $\gamma_0 = 0{\cdot}01$, $\varepsilon = 0{\cdot}05$, $v = 0{\cdot}3$, $l_y/l_x = 1, h/l_x = 0{\cdot}01$, $\hat{Q} = 1$. Curve 1 is plotted for $\hat{\gamma}_1 = 0$, i.e. the case of an uncontrolled plate. It has a sharp resonant peak at the frequency $\omega_s h/c_p = 0{\cdot}0061$. However, if the parametric stiffness modulation is performed (the modulation amplitude is $\hat{\gamma}_1 = 0{\cdot}1\gamma_0 = 0{\cdot}001$, the stiffness modulation frequency equals the third resonant frequency of dominantly shear vibrations, $\omega_f = \omega_{322}$), then (see curve 2 in Figure 5) the resonant peak occurs at much higher frequency ($\omega_s h/c_p = 0{\cdot}0139$). Similarly, if a plate with such a stiffness modulation is excited in the frequency range close to $\omega_s h/c_p = 0{\cdot}0061$ (the resonant frequency of a plate with $\hat{\gamma}_1 = 0$), then the amplitude of forced vibrations W_{s11} is much smaller than in the uncontrolled case. The same holds true at $\omega_s h/c_p = 0{\cdot}0061$ for the two other "slow" amplitudes Θ_{sx11} and Θ_{sy11}, but they are not too important because (as seen from Figure 2(b)) this mode is of the dominantly flexural type. Naturally, at a frequency of $\omega_s h/c_p = 0{\cdot}0139$ all three amplitudes have resonant peaks. On the other hand, at a frequency of $\omega_s h/c_p = 0{\cdot}0061$ the amplitudes of "fast" vibrations of a controlled plate (i.e., $\hat{\gamma}_1 = 0{\cdot}1\gamma_0 = 0{\cdot}001$) are larger, than in the case of the absence of control, $\hat{\gamma}_1 = 0$ (when they are negligibly small). However, the amplitude of the flexural component W_{f22} does not exceed the amplitude W_{s11} so that flexural vibrations of a plate with parametric stiffness modulation are generally suppressed at this frequency. The amplitudes of shear components Θ_{sy11} and Θ_{sx11} grow more substantially, but from the viewpoint of sound radiation, in-plane vibrations are not of much practical importance. The issue of the sound radiation control of a plate with parametric stiffness modulation will be considered in section 6.

5. EIGENFREQUENCIES OF VIBRATIONS OF A RECTANGULAR SANDWICH PLATE WITH PARAMETRIC STIFFNESS MODULATION

In the previous section, the problem of forced vibrations has been solved for a plate with the stiffness modulation. As shown, a plate with a controlled stiffness still exhibits the resonant behaviour, but the resonant peak is shifted towards larger values of the excitation frequency. In fact, a resonant growth of the amplitude of forced vibrations is associated with the determinant of equations (26) tending to zero. Therefore, the problem of "free vibrations" may be posed for a plate with the parametric stiffness modulations. Note that this is not a "standard" formulation of the problem in free vibrations, since some forced motions are already "embedded" into the mechanical system and it has a reduced number of degrees of freedom. However, it is possible to identify values of the "slow" excitation frequencies at which solution of linear algebraic equations (26) is unbounded as the eigenfrequencies.

In such a formulation, parameters of a sandwich plate composition, which control these eigenfrequencies should include (besides the physical parameters of thickness, stiffness and density) the parameters of "vibrational rheology", see references [3–4, 7]. These parameters specify "hidden" motions, which produce stiffness modulation. They are the frequency of stiffness modulations ω_f and the amplitude of modulation $\hat{\gamma}_1$. Similarly, the "slow" resonant frequency ω_s becomes dependent on both these parameters. To formulate this dependence, the following determinant should be put to zero:

$$\begin{pmatrix} a_{11}-b_1\omega_s^2 & a_{12} & a_{13} & \hat{\gamma}_1 a_{14} & \hat{\gamma}_1 a_{15} & \hat{\gamma}_1 a_{16} \\ a_{21} & a_{22}-b_2\omega_s^2 & a_{23} & \hat{\gamma}_1 a_{24} & \hat{\gamma}_1 a_{25} & \hat{\gamma}_1 a_{26} \\ a_{31} & a_{32} & a_{33}-b_3\omega_s^2 & \hat{\gamma}_1 a_{34} & \hat{\gamma}_1 a_{35} & \hat{\gamma}_1 a_{23} \\ \hat{\gamma}_1 a_{41} & \hat{\gamma}_1 a_{42} & \hat{\gamma}_1 a_{43} & a_{44}-b_4\omega_f^2 & a_{45} & a_{46} \\ \hat{\gamma}_1 a_{51} & \hat{\gamma}_1 a_{52} & \hat{\gamma}_1 a_{53} & a_{54} & a_{55-}b_5\omega_f^2 & a_{56} \\ \hat{\gamma}_1 a_{61} & \hat{\gamma}_1 a_{62} & \hat{\gamma}_1 a_{63} & a_{64} & a_{65} & a_{66}-b_6\omega_f^2 \end{pmatrix} = 0. \quad (27)$$

As is clearly seen from equation (27), the modal interaction vanishes as soon as $\hat{\gamma}_1 = 0$. Then the resonant frequencies of slow motions simply coincide with those of vibrations of a plate $(k = 1, m = 1)$ without the stiffness modulation.

The curves in Figure 6 display a dependence of the first "slow" resonant frequency on the stiffness modulation amplitude $\hat{\gamma}_1$ for several values of the detuning parameter β, which is introduced as $\omega_f = \beta\omega_{223}$. The parameters of composition of a sandwich plate are the following: $\delta = 0{\cdot}1$, $\gamma_0 = 0{\cdot}01$, $\varepsilon = 0{\cdot}05$, $v = 0{\cdot}3$, $l_y/l_x = 1$, $h/l_x = 0{\cdot}01$. Curve 1 is plotted for $\beta = 1{\cdot}000001$, curves 2, 3 and 4 are plotted for $\beta = 1{\cdot}0001, 1{\cdot}001$ and $\beta = 1{\cdot}01$ respectively. As seen, the closer a value of the detuning parameter β is to 1, the faster is the growth in the controlled resonant frequency. Apparently, the most efficient control is achieved in the case of a perfect tuning of the stiffness modulation frequency to the "fast" resonant one. As it is also seen clearly from these figures, the suggested mechanism of control is very sensitive to deviations of the modulation frequency from its "nominal" value ω_{322}. For all other parameters of a sandwich plate composition these curves remain qualitatively the same. However, the sensitivity to deviations in detuning parameter decreases when the length parameter l_y/l_x grows. In fact, starting from $l_y/l_x = 5$, results for a rectangular plate almost merge with results reported in reference [3] for a plate in one–dimensional cylindrical bending.

6. THE INFLUENCE OF INTERNAL DAMPING AND OF AN ACOUSTIC MEDIUM ON FORCED VIBRATIONS OF A CONTROLLED PLATE

In the previous sections, vibrations of a sandwich plate have been considered provided that no energy dissipation occurs in the material of all plies. The forced vibration of a structure with some material losses is now addressed. In the framework of present analysis

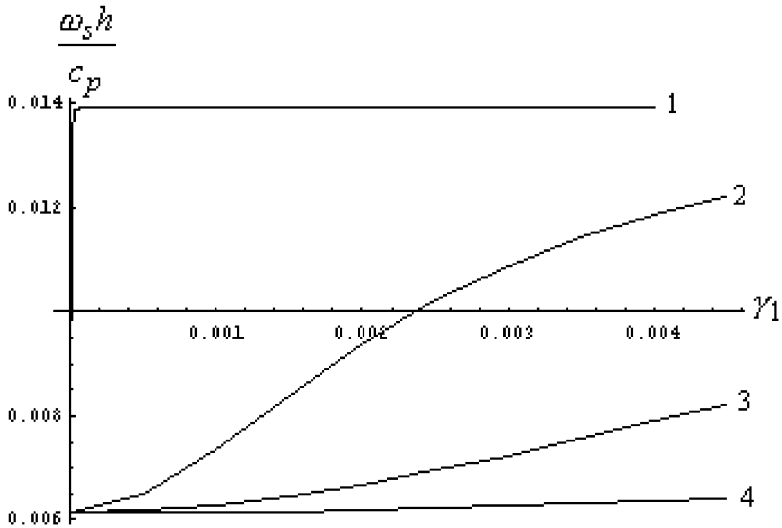

Figure 6. The first resonant frequency versus the amplitude of the stiffness modulation.

we use the simple model of a visco-elastic behaviour of the material for both the core and the skin plies. Since harmonic vibrations are considered, the real-valued Young's modulae E_s and $E_c = \gamma E_s$ in equations (11) are replaced by $E_s(1 - \mathrm{i}\chi)$ and $E_c = E_c\gamma(1 - \mathrm{i}\chi)$ respectively. The parameter χ in these formulas characterizes the viscous properties of the material of plies and therefore defines the energy dissipation there. Here for the sake of simplicity the energy dissipation will be assumed to be the same in all plies. Note that a detailed analysis of the role of damping in the skin plies and in the core has been performed in reference [4].

To find a forced response (a spatial distribution of the driving force is given by equation (22)), an inhomogeneous system of linear algebraic equations (26) is solved. The curves shown in Figure 7 indicate a dependence of the amplitude of forced vibrations of a plate $|W_{s11}| \equiv |A_{11}|$on the frequency parameter $\omega_s h/c_p$ with the energy losses taken into account. The parameters of a plate are $\delta = 0{\cdot}1$, $\gamma_0 = 0{\cdot}01$, $\varepsilon = 0{\cdot}05$, $v = 0{\cdot}3$, $l_y/l_x = 1$, $h/l_x = 0{\cdot}01$. The parameters of excitation are $\hat{Q} = 1$, $\beta = 1{\cdot}000001$. Curve 1 presents the amplitude of vibrations of a plate without control and without damping, $\hat{\gamma}_1 = 0, \chi = 0$. This curve has a resonant peak at $\omega_s h/c_p = 0{\cdot}0061$. Curve 2 is plotted for a controlled plate without damping ($\hat{\gamma}_1 = 0{\cdot}001, \chi = 0$). This curve has a resonant peak at $\omega_s h/c_p = 0{\cdot}0139$. Curve 3 is plotted for $\hat{\gamma}_1 = 0{\cdot}001, \chi = 0{\cdot}000001$. It still has the resonant peak at $\omega_s h/c_p = 0{\cdot}0139$, but this peak is bounded due to the presence of the energy dissipation. Curve 4 is plotted for $\hat{\gamma}_1 = 0{\cdot}001, \chi = 0{\cdot}00001$ and the resonant peak is still at the same frequency as in the previous two cases, but it is heavily damped. Finally, curve 5 is plotted for $\hat{\gamma}_1 = 0{\cdot}001, \chi = 0{\cdot}0001$. Although vibrations are heavily damped, the resonant peak is shifted towards its "initial" position, i.e., to $\omega_s h/c_p = 0{\cdot}0061$. Inspection into the role of an internal damping shows that similar to a standard case of forced vibrations of an uncontrolled structure, material losses reduce the amplitude of vibrations near resonant excitation and they produce a phase shift between displacement and a driving force. However, there is also another aspect of the presence of material losses. As discussed, to maximize an efficiency of the parametric stiffness modulation, its frequency should be as close as possible to the eigenfrequency ω_{322}. The latter actually becomes complex valued as

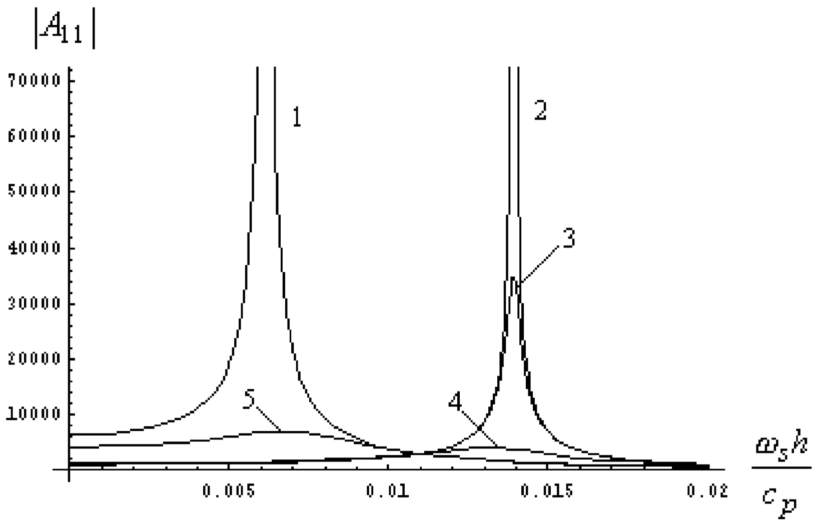

Figure 7. The amplitude W_{s11} versus frequency parameter $\omega_s h/c_p$: the influence of damping.

soon as $\chi \neq 0$. When the stiffness modulation is initiated, its time dependence is purely periodic, i.e., it is expressed by a real-valued frequency. Thus, it occurs somewhat out of tune with the exact value of the eigenfrequency of a plate with material losses and the efficiency of vibration suppression due to modal interaction drops down.

As is well known, the "energy leakage" from a vibrating plate may be associated either with internal losses in plate"s material or with an external damping due to the sound radiation. It should be pointed out that once structural acoustic interaction is taken into account, then the problem of vibration control merges with the problem of control of the sound radiation. Thus, the suggested method of suppression of vibrations is naturally extended to sound radiation control, which is very important in many practical applications. To the best of our knowledge, the problem of active control of sound radiation from elastic structures has been considered in a coupled formulation related to the so-called "heavy fluid loading" only in references [3, 4]. An uncoupled formulation of the problem of control of sound radiation (applicable in "light fluid loading conditions") has been analyzed by many authors, see the list in reference [1]. Note that modern composite materials (in particular, materials of a sandwich type) are characterized as essentially light-weighted ones [16]. Thus, it is reasonable to suggest that it is necessary to analyze their interaction with an acoustic medium within the framework of a theory of structural dynamics in heavy fluid loading conditions.

A coupled formulation of the problem in stationary structural acoustics is now explored. Then due to the linearity of formulation of an acoustical problem, the Helmholtz equation is written individually for "fast" and "slow" motions:

$$\Delta\varphi_s + \left(\frac{\omega_s}{c_f}\right)^2 \varphi_s = 0, \quad \Delta\varphi_f + \left(\frac{\omega_f}{c_{fl}}\right)^2 \varphi_f = 0. \tag{28}$$

In equation (28), φ_s, φ_f are velocity potentials generated by the "slow" and the "fast" motions of a plate, respectively, c_{fl} is the sound speed in an acoustic medium.

These equations should be solved simultaneously with the equations of structural dynamics (17) and (21), which contain the fluid loading term, presenting the contact acoustic pressure developed in "fast" and "slow" motions, respectively (ρ_{fl} is the density of an acoustic medium):

$$p_s = \mathrm{i}\rho_{fl}\omega_s\varphi_s, \quad p_f = \mathrm{i}\rho_{fl}\omega_f\varphi_f. \tag{29}$$

The formulation of structural–acoustic coupling is completed by the continuity conditions for velocities of a plate and an acoustic medium at the fluid–structure interface. A model of an elastic plate put into an infinitely long rigid baffle is used. An acoustic medium occupies the lower half-space. z is the Cartesian co-ordinate perpendicular to the surface of a plate and directed outside the fluid''s volume. Thus, these conditions are written as

$$\frac{\partial\varphi_s}{\partial z} = -\mathrm{i}\omega_s w_s, \quad \frac{\partial\varphi_f}{\partial z} = -\mathrm{i}\omega_f w_f. \tag{30}$$

As is well known [17], solution of the problem in acoustics in this case gives the formulation of a contact pressure in the form of a Rayleigh integral. The presence of an acoustic medium does not affect the formulation of the method of direct partition of motions and in the equations of "slow" and "fast" motions the following terms should be added:

$$p_s(x,y) = \frac{\rho_{fl}\omega_s^2}{2\pi}\int_0^{l_x}\int_0^{l_y} w_s(\xi,\eta)\,\frac{\exp(\mathrm{i}(\omega_s/c_{fl})\sqrt{(x-\xi)^2+(y-\eta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

$$p_f(x,y) = \frac{\rho_{fl}\omega_f^2}{2\pi}\int_0^{l_x}\int_0^{l_y} w_f(\xi,\eta)\,\frac{\exp(\mathrm{i}(\omega_f/c_{fl})\sqrt{(x-\xi)^2+(y-\eta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{31}$$

Thus, the diagonal elements b_1 and b_4 of matrix (27) are complemented by the following integrals ($k = 1, 2$, $n \equiv s,f$):

$$\int_0^{l_x}\int_0^{l_y}\int_0^{l_x}\int_0^{l_y} \sin\left(\frac{k\pi x}{l_x}\right)\sin\left(\frac{k\pi\xi}{l_x}\right)\sin\left(\frac{k\pi y}{l_y}\right)\sin\left(\frac{k\pi\eta}{l_y}\right)$$
$$\frac{\exp(\mathrm{i}(\omega_n/c_{fl})\sqrt{(x-\xi)^2+(y-\eta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\eta\,\mathrm{d}y.$$

The real parts of these integrals contribute to the fluid added mass, their imaginary parts are related to radiation damping.

Fluid loading is specified by two parameters, which in the present calculations are taken as $\rho_{fl}/\rho_p = 0{\cdot}128$ and $c_{fl}/c_p = 0{\cdot}307$. They are relevant to vibrations of a sandwich plate with steel skin plies in water. Similar to the case of vibrations of a plate with material losses, the presence of acoustic damping results in transformation of purely real-valued eigenfrequencies to complex-valued ones. However, as is well known [17] at the low-frequency range the real part of an eigenfrequency of free vibrations is very close to a relevant resonant frequency of forced vibrations. In Figure 8(a), a dependence of the first resonant frequency parameter of a plate on l_x/h is shown for vibrations in a vacuum (upper curve) and in an acoustic medium (lower curve). This branch is relevant to the low-frequency dominantly flexural vibrations. As is seen, the role of added mass is very important and it grows with an increase in the parameter l_x/h. The parameters of a plate are $\delta = 0{\cdot}1$, $\gamma_0 = 0{\cdot}01$, $\varepsilon = 0{\cdot}05$, $\nu = 0{\cdot}3$, $l_y/l_x = 1$. However, the presence of acoustic medium does not affect eigenfrequencies of dominantly shear vibrations, as seen in Figure 8(b), where the branch relevant to vibrations of this type is shown. This phenomenon is explained by the absence of viscosity in a fluid. In such a case, a fluid does

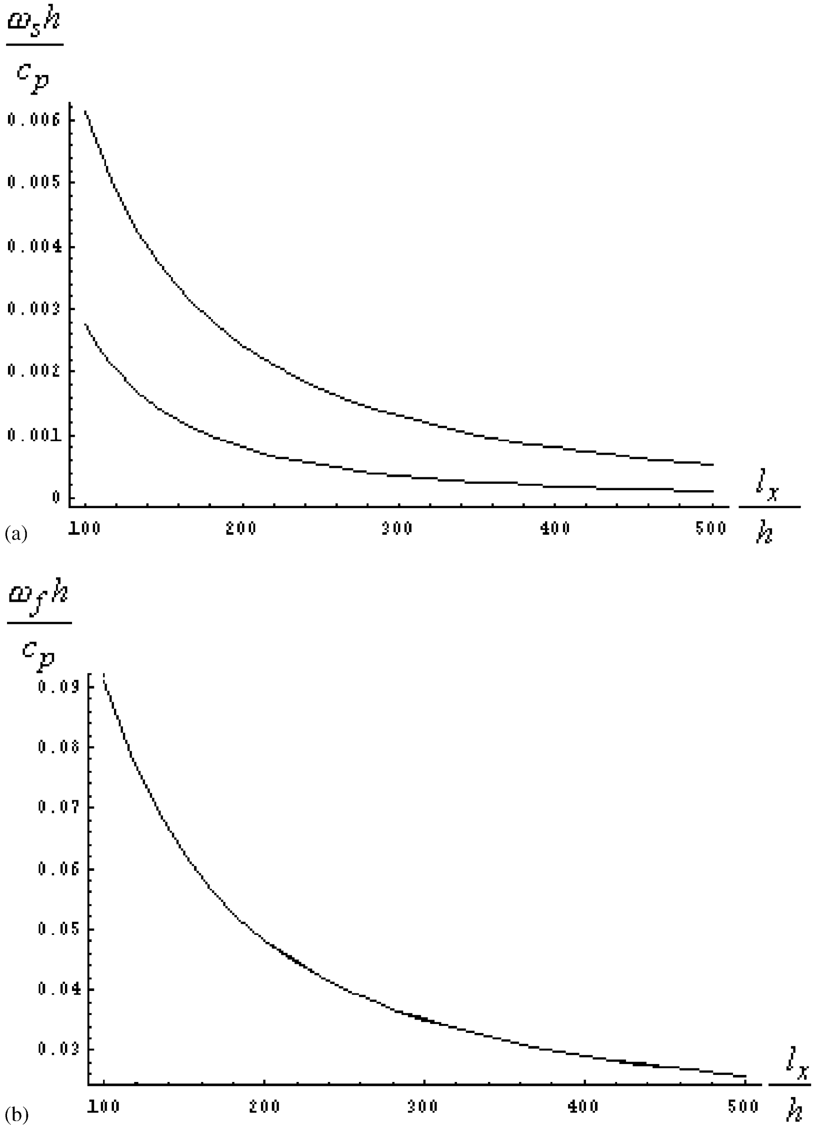

Figure 8. (a) The eigenfrequency parameter $\omega_{111}h/c_p$ of a square plate with and without fluid loading versus l_x/h. (b) The eigenfrequency parameter $\omega_{223}h/c_p$of a square plate with and without fluid loading versus l_x/h.

not respond to dominantly in-plane motions of a plate, which are associated with shear deformations.

The mechanism of active control of vibrations is not affected by the presence of an acoustic medium and it is illustrated by the graph shown in Figure 9. The ratio $\omega_{111}|_{\hat{\gamma}_1 \neq 0}/\omega_{111}|_{\hat{\gamma}_1 = 0}$ is chosen as a convenient measure of the efficiency of the stiffness modulation. In the case of absence of the stiffness modulation, the ratio of the first eigenfrequency of a controlled plate to the first eigenfrequency of an uncontrolled plate equals 1, as designated by point 1 in this graph. In the case of a perfect tuning of the stiffness modulation frequency ω_f to the eigenfrequency ω_{223} ($\hat{\gamma}_1 = 0.001, \beta = 1$), the effect

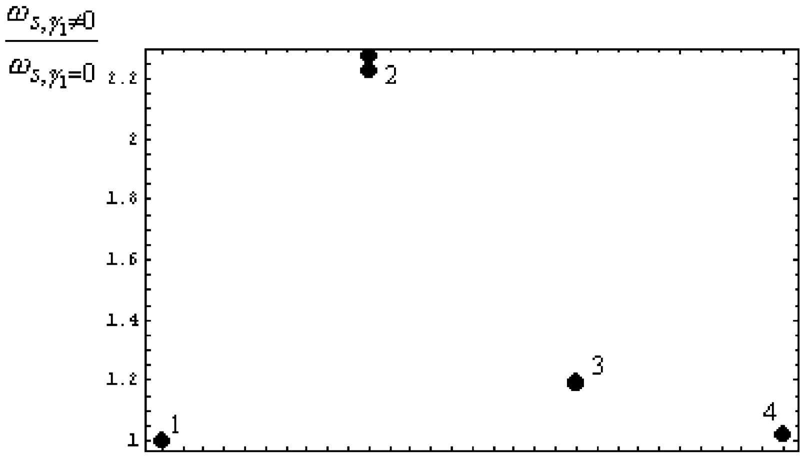

Figure 9. The comparison of control efficiency with and without fluid loading.

of the stiffness modulation in a fluid is just very slightly less than in vacuum. Point 2 designated by circle for a plate without fluid loading lies a little above the point designated by rectangle for a plate with fluid loading. In the case of $\hat{\gamma}_1 = 0{\cdot}001$, $\beta = 1{\cdot}001$, the efficiency in a fluid and in vacuum is exactly the same (at point 3 circle coincides with rectangle) and the increase in the eigenfrequency is about 20%. In the case of $\hat{\gamma}_1 = 0{\cdot}001$, $\beta = 1{\cdot}01$, the efficiency in a fluid and in vacuum is exactly the same (at point 4, circle also coincides with rectangle), but it drops to $\omega_{111}|_{\hat{\gamma}_1 \neq 0}/\omega_{111}|_{\hat{\gamma}_1=0} = 1$ (in effect, the mechanism of control does not work). For all other parameters of a sandwich plate composition, the roles of an internal damping and of an acoustic medium remain qualitatively the same. For $l_y/l_x > 5$, results obtained for a rectangular plate merge with the results obtained for a beam (a plate in one-dimensional cylindrical bending) reported in reference [3].

7. DISCUSSION OF THE PARAMETRIC STIFFNESS MODULATION

As is shown in previous sections of the paper, the parametric stiffness modulation may be used as a tool to control resonant vibrations of sandwich plates. However, before summing up the results reported in the previous sections, it is appropriate to formulate some remarks concerning several aspects of this way of an active control of vibrations and the method used to solve this problem.

So far, the possible types of a micro-structure, which may be suggested to provide the necessary stiffness modulations have not been discussed. This aspect (which is very important and possibly crucial for a practical applicability of this method) has been tackled in references [2, 4]. As mentioned in reference [2], the effective stiffness of the plate can be changed by varying the internal structure of the core material or by having multiple pin-like inclusions whose orientation may be varied. Switched elements [18] based on piezoceramic micro-actuators [19] shape memory alloys, etc. also offer a wide opportunity for distributed stiffness control. The control of vibrations at comparatively low frequencies of a structure and wavelengths pronouncedly exceeding the characteristic size of a micro-structure are considered. The frequency of the modulatory signal is markedly higher, because it is associated with rather small deviations of initial stiffness parameters.

However, from the practical viewpoint, the necessary modulation frequency is not too high (up to 200 KHz) and lies within the working frequency range of piezoelectric actuators, which may be inserted into the micro-inhomogeneous material.

Apparently, an active control of any kind implies at additional energy is input into the structure. Then it is also of high relevance to compare this input with, for example, a decrease in the energy of vibrating structure due to a decrease in forced amplitudes of vibrations. Of course, such a comparison may make sense only when a particular mechanism of the stiffness modulation at the micro-level is specified. Although this important aspect has not been tackled in this paper, it has been thoroughly analyzed in the related paper [4] for a honeycomb beam (a plate in one-dimensional cylindrical bending). Results reported in this paper with respect to this energy balance show that for precise tuning, the energy input is much less than reduction in the energy of slow vibrations, so that this mechanism of suppression of vibrations is efficient.

As is already mentioned, the asymptotic predictions obtained in this paper by the method of direct partition of motions for a rectangular sandwich plate are in perfect agreement with similar results presented in references [3, 4] for vibrations of a sandwich plate in a one-dimensional cylindrical bending. In reference [4], direct numerical integration of equations similar to equations (17) and (21) has been made and steady state amplitudes of forced vibrations of a controlled honeycomb beam have coincided with the predictions obtained by the method of direct partition of motions. The results of such a comparison in the case considered here are similar and we do not report them here in detail for brevity.

It should also be pointed out that the suggested control mechanism is applicable for a sandwich rectangular plate with any boundary conditions. The structure of frequency spectra found in section 2 for Navier conditions remains the same for other boundary conditions. Thus, the eigenfrequencies of dominantly shear vibrations are markedly higher than the first eigenfrequency of dominantly flexural vibrations and the "fast" stiffness modulations may control "slow" forced resonant motions. Similarly, the selection of a spatial distribution of the stiffness modulation should be adjusted at each particular case to provide a necessary link between two interacting modes.

8. THE ILLUSTRATIVE ELEMENTARY TWO-DEGREE-OF-FREEDOM MODEL

As has been discussed, the stiffness modulation generates the modal interaction between near-resonant "slow" motions at the excitation frequency and near-resonant "fast" motions at the modulation frequency. An elementary two-degree-of-freedom linear mechanical system shown in Figure 10 exhibits a similar behaviour in the resonant excitation conditions. It consists of two masses M_0 and m linked to some foundation by the springs of stiffness K_0, K_m and connected to each other by a spring of stiffness K_Δ. For simplicity, it is assumed that the energy dissipations in both elements are the same and characterized by the parameter D_0. The governing equations of motions of such a system driven by a force $\bar{Q}$ (which acts at the mass M_0) are obvious:

$$M_0 \frac{\mathrm{d}^2 x_0}{\mathrm{d}t^2} = \bar{Q} - D_0 \frac{\mathrm{d}x_0}{\mathrm{d}t} - K_0 x_0 - K_\Delta (x_0 + x_m), \tag{32a}$$

$$m \frac{\mathrm{d}^2 x_m}{\mathrm{d}t^2} = -D_0 \frac{\mathrm{d}x_m}{\mathrm{d}t} - K_m x_m - K_\Delta (x_0 + x_m). \tag{32b}$$

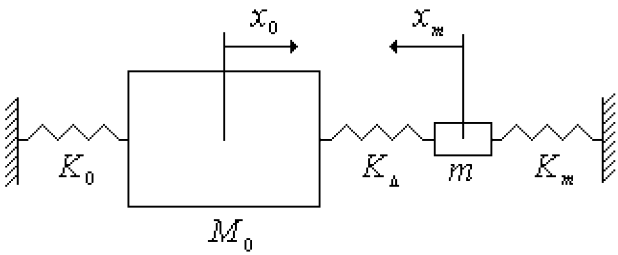

Figure 10. The model two-degree-of-freedom system.

Clearly, by letting $K_\Delta = 0$ two linear uncoupled oscillators are obtained exactly as the two modes of vibrations of a sandwich plate become uncoupled when $\bar{\gamma}_1 = 0$. Furthermore, it is assumed ($\gamma_* \ll 1$)

$$K_0 = K - K\gamma_* \cos \omega_f t, \quad K_m = K - K\gamma_* \cos \omega_f t, \quad K_\Delta = K\gamma_* \cos \omega_f t. \tag{33}$$

The springs K_0, K_m have modulated stiffness, whereas the "spring" between two masses changes its stiffness periodically, i.e., it exhibits a negative reactive force as well as a positive one. A practical implementation of such a device for a two-degree-of-freedom model is not discussed in detail, but similar devices are often introduced in a control theory. If a harmonic resonant driving force $\bar{Q} = Q \cos \omega_s t$ acts at mass M_0 at frequency $\omega_s = \omega_1 + \sigma_s (\omega_1 = \sqrt{K/M_0})$, and if the modulation frequency is $\omega_f \approx \omega_2 = \sqrt{K//m} \gg \omega_s$ ($\omega_f = \omega_2 + \sigma_f$), then the set of equations (32) is conveniently solved by the method of multiple scales [14]. Solution is sought (to the zeroth order) as ($T_0 = t, T_1 = \varepsilon_0 t$)

$$\begin{aligned} x_{00} &= A(T_1)\exp(\mathrm{i}\omega_1 T_0) + \bar{A}(T_1)\exp(-\mathrm{i}\omega_1 T_0), \\ x_{m0} &= B(T_1)\exp(\mathrm{i}\omega_2 T_0) + \bar{B}(T_1)\exp(-\mathrm{i}\omega_2 T_0). \end{aligned} \tag{34}$$

In equations (34), standard substitution is used

$$A(T_1) = a(T_1)\exp(\mathrm{i}\hat{\varphi}(T_1)) \quad B(T_1) = b(T_1)\exp(\mathrm{i}\psi(T_1))$$

and the phase angles are formulated as

$$\Phi = \hat{\varphi} - \sigma_s T_1, \quad \Psi = \psi - \sigma_f T_1.$$

The standard technique [14] gives the following solution:

$$a = \frac{1}{2}\frac{Q}{K}\left\{\left[2\hat{\sigma}_s - \frac{\gamma_*^2(\hat{\sigma}_s + \hat{\sigma}_f)\sqrt{\mu}}{4F^2}\right]^2 + \bar{\chi}^2\left[1 + \frac{\gamma_*^2}{4F^2\sqrt{\mu}}\right]^2\right\}^{-1/2}, \quad b = \frac{\gamma_* a}{2F}, \tag{35a, b}$$

$$\tan \Psi = \frac{\chi}{\mu(\hat{\sigma}_s + \hat{\sigma}_f)}, \quad \tan \Phi = \frac{\bar{\chi}(1 + \gamma_*^2/4F^2\sqrt{\mu})}{2\hat{\sigma}_s - \gamma_*^2\sqrt{\mu}(\hat{\sigma}_s + \hat{\sigma}_f)/4F^2}, \tag{35c, d}$$

$$F \equiv \sqrt{\frac{\bar{\chi}^2}{\mu} + \mu(\hat{\sigma}_s + \hat{\sigma}_f)^2}, \quad \mu = m/M_0, \quad \bar{\chi} = D_0/\omega_1 M_0, \quad \hat{\sigma}_s = \sigma_s/\omega_1, \quad \hat{\sigma}_f = \sigma_f/\omega_1. \tag{35e}$$

To judge upon stability of this solution, a Jacobean is set-up and its eigenvalues are computed for the amplitudes and phases given by formalas (35a–d). For any realistic combinations of parameters, all eigenvalues have negative real parts so that this solution is stable.

Now the method of direct partition of motions is applied, and for simplicity consider an undamped system, i.e., put $D_0 = 0$ in equations (32). Motions of mass m are sought as $x_m = \tilde{x}_m(t)\cos\omega_f t$ and substitution of this formula into equation (32b) gives

$$(K - m\omega_f^2)\tilde{x}_m = -\gamma_* K x_0 \tag{36}$$

so that equation (32a) is transformed as

$$M\frac{\mathrm{d}^2 x_0}{\mathrm{d}t^2} + K\left[1 - \frac{\gamma_*^2\cos^2\omega_f t}{1 - (\omega_f/\omega_2)^2}\right]x_0 = Q\cos\omega_s t. \tag{37}$$

Assume that $x_0 = \hat{x}_0\cos\omega_s t,\ \tilde{x}_m = \hat{x}_m\cos\omega_s t,\ \omega_s \ll \omega_f$, so that as suggested by the method of direct partition of motions the averaging over a period of "fast" motions gives $\langle\cos^2\omega_f t\rangle = \frac{1}{2}$. Then equation (36) is reduced to a simple form

$$\left[1 - \left(\frac{\omega_s}{\omega_1}\right)^2 - \frac{1}{2}\frac{\gamma_*^2}{1 - (\omega_f/\omega_2)^2}\right]\hat{x}_0 = \frac{Q}{K}. \tag{38}$$

If notation (35e) is introduced and not that $\mu = m/M_0 \equiv (\omega_0/\omega_m)^2$, this equation becomes

$$\left[1 - (1 + \hat{\sigma}_s)^2 - \frac{1}{2}\frac{\gamma_*^2}{1 - (1 + \hat{\sigma}_f\sqrt{\mu})^2}\right]\hat{x}_0 = \frac{Q}{K}. \tag{39}$$

Furthermore, if it is assumed that vibrations are excited precisely at the frequency $\omega_s = \omega_1$ and only the first term in expansion of the last term in equation (39) is retained, one gets

$$\hat{x}_0 = \frac{4\hat{\sigma}_f\sqrt{\mu}}{\gamma_*^2}\frac{Q}{K}, \quad \hat{x}_m = \frac{2Q}{\gamma_*}. \tag{40a, b}$$

It is twice as large as the predictions for this case given by formulas (35a,b). However, these results indeed match each other perfectly, since the full forced response determined by the method of multiple scales is two times larger equation (35a,b), see its full formulation (34), which includes both $A(T_1), B(T_1)$ and their complex conjugates.

Formula (40a) shows that the amplitude $\hat{x}_0$ of the resonantly directly excited mass M_0 tends to zero if so does the detuning parameter of modulation $\hat{\sigma}_f$. As is shown in section 6, in the case of a sandwich plate, the behaviour of the amplitude W_{s11} is similar. However, for a sandwich plate, it is not possible to eliminate completely vibrations of the directly excited mode, whereas vibrations of mass M_0 may be completely stopped in the model two-degree-of-freedom system. On the other hand, as is seen from formula (40b), the amplitude of vibrations of the second mass sharply grows in the model system, whereas in the sandwich place similar effect is less pronounced and is associated with in-plane shear motions.

Of course, the behaviour of a rectangular sandwich plate even in Galerkin four-term approximation (23) is more complicated than the behaviour of the model system shown in Figure 10. However, dynamics of this model system explains to some extent the mechanism of an active control of vibrations of sandwich plates by the parametric stiffness modulation.

9. CONCLUSIONS

A theoretical analysis is performed for active control of vibrations of a rectangular sandwich plate. Equations of dynamics of such a plate are derived as stationarity conditions for the Hamiltonian of a plate. In an elementary case of the constant stiffness parameters, three frequency spectra of free vibrations are found for a simply supported plate. The first one is relevant to dominantly flexural low-frequency vibrations, the other two are associated with high-frequency dominantly shear motions.

An active control of forced vibrations is performed by the parametric stiffness modulation. It is shown that this kind of an active control is able to significantly reduce resonant forced response of a plate. This effect is achievable by the fairly small amplitude of the stiffness modulation, if it is imposed at the resonant frequency of dominantly shear vibrations of an uncontrolled plate. The efficiency of the suggested mechanism of control is shown to be sensitive to the detuning parameter, which quantifies the difference between the stiffness modulation frequency and the eigenfrequency of shear vibrations of an uncontrolled plate. However, this control mechanism is not sensitive to the other parameters of a rectangular plate, e.g., $\varepsilon, \gamma_0, \delta, l_y/l_x$. An influence of energy losses in plate's material and of radiation damping on the efficiency of the suggested control mechanism are also studied. Generally, the presence of moderate losses in the material of a plate does not destroy the efficiency of the control of forced vibrations shown for an ideally elastic material. This also holds true with respect to the role of an acoustic medium. Dynamics of an elementary two-degree-of-freedom model mechanical system is considered to illustrate the mechanism of modal interaction, which is involved in the suggested way of an active control of vibrations of sandwich plates. It is shown that the method of direct partition of motions and the method of multiple scales give identical results in the case, when the controlled resonant frequency is much lower than the frequency of the stiffness modulation.

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